Lecture 6: Rational Surfaces

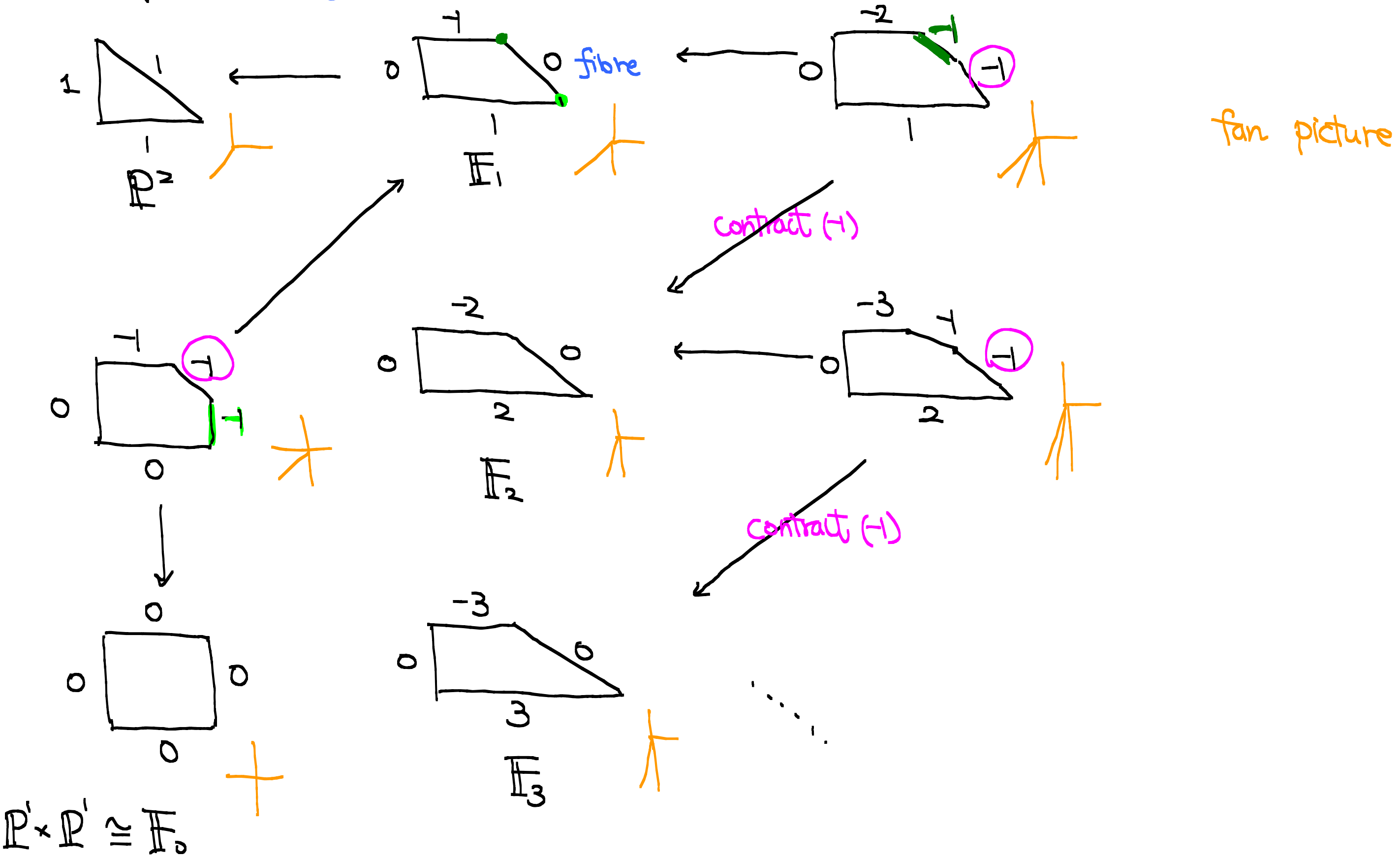
Note Title 8/25/2020

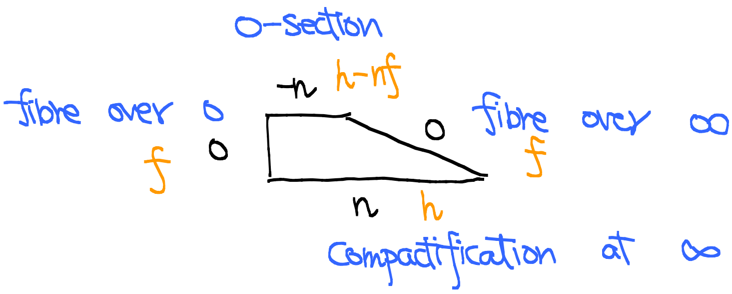
Definition: A surface X is <u>rational</u> if it is birational to $\mathbb{P}^2$.

ex. Hirzebruch surfaces $\mathbb{F}_n := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)) \longrightarrow \mathbb{P}^1$

the only ruled surface / $\mathbb{P}^1$

toric picture   cycle of anti-canonical divisor





$\mathbb{F}_n$ viewed as compactification of total space of $\mathcal{O}_{\mathbb{P}^1}(-n)$

Proposition 1. $\mathbb{F}_n \ncong \mathbb{F}_m$ unless $n=m$.

$\mathbb{F}_n$ is minimal except $n=1$.

pf: Recall that $\mathrm{Pic}(\mathbb{F}_n) = \underset{\text{fibre}}{\mathbb{Z}f} \oplus \underset{\mathcal{O}(1)}{\mathbb{Z}h}$

$$\text{w/ } f.h = 1, \quad f^2 = 0, \quad h^2 = n \geq 0$$

Claim: the zero section is the only negative self-intersection curve

Assume that $af + bh$ effective, not the zero section

$$\xrightarrow[|f| \text{ no base point}]{} (af+bh).f \geq 0 \qquad \therefore\ b \geq 0$$

$$\xrightarrow[\substack{af+bh \text{ not} \\ \text{the zero section}}]{} 0 \leq (af+bh).(h-nf) = a - nb + bn \qquad \therefore\ a \geq 0$$

$$\therefore\ (af+bh)^2 = 2ab + b^2 n = b(2a+bn) \geq 0$$

Remark: $\mathbb{F}_n, \mathbb{F}_m$ are deformation equivalent, if $n \equiv m \pmod 2$

In particular, only 2 diffeomorphism types.

ex.

$$\begin{array}{ccc} \mathcal{X} & & \text{small resolution at } (0,0,0,1) \\ \downarrow & & \\ \{x_0x_1 = x_2^2 - t^2x_3^2\} & \subseteq & \mathbb{CP}^3_{(x_0,x_1,x_2,x_3)} \times \mathbb{C}_t \\ & \searrow & \downarrow \\ & & \mathbb{C}_t \end{array}$$

X_t fibres $\hookrightarrow \{x_0x_1 = x_2^2 - t^2x_3^2\}$

$t \neq 0$. X_t smooth quadric in $\mathbb{P}^3$, $\cong \mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_0$

$t = 0$. X_0 blow up of $\underset{\text{Cone over } \mathbb{C}^2/\mathbb{Z}_2}{\{x_0x_1 = x_2^2\}}$ at $(0,0,0,1)$

$X_0 \cong \mathbb{F}_2$

ex. del Pezzo surface revisit

X_d: del Pezzo surface of degree

$= $ blow up of $\mathbb{P}^2$ at generic $9-d$ points

Otherwise $X_d \xrightarrow{|-K_{X_d}|} \mathbb{P}^d$ not embedding

$-K_{X_d}$ not ample

no 3 of them colinear
6 of them lie on a conic
8 of them lie on a cubic
w/ a node on one of them

- $\mathrm{Aut}(\mathbb{P}^2) \cong PGL(3) \Longrightarrow X_9, X_8, X_7, X_6, X_5$ are unique

- X_9, X_8, X_7, X_6 are toric.

Proposition 2: For $0 \le 9-d \le 6$, $X_d \overset{|-K_{X_d}|}{\hookrightarrow} \mathbb{P}^d$

In particular, $X_3 \subseteq \mathbb{P}^3$ is a cubic surface

$X_4 \subseteq \mathbb{P}^4$ is intersection of 2 quadrics.

pf: $|-K_{X_d}|$ defines an embedding

$\Longleftrightarrow$ ① separate points
② separate tangents

only need to check the case $d=3$

$$\begin{array}{ll} X_3 = Bl_{p_1,\cdots,p_6}\mathbb{P}^2 \ni \pi(x) \notin \{p_1,\cdots,p_6\} & \\ \pi \downarrow & \\ \mathbb{P}^2 & \end{array}$$

$\rightsquigarrow$ $\exists!$ Conic Q_{ij}^x passing through x & $p_{k\neq i,j}$

$\exists!$ Conic Q_i passing through $p_{k\neq i}$

$\pi^{-1} Q_i \cap \pi^{-1} Q_j = \phi$ if $i \neq j$

① $x \neq y \in X_3$, $p_i \neq \pi(x), \pi(y)$, $x \notin \pi^{-1} Q_i$

- $\pi^{-1} Q^x_{ij} \cap \pi^{-1} Q^x_{ik} = \{x\}$ $\quad p_k \neq p_i, p_j, \pi(x)$

$$\Downarrow$$

$$y \in \pi^{-1} Q^x_{ik} \text{ for at most one } k$$

Also $y \in \pi^{-1}(\overline{p_i p_k})$ for at most one k

$\therefore$ $\exists k$ s.t. $x \in \pi^{-1} Q^x_{ik} \cup \pi^{-1}(\overline{p_i p_k}) \not\ni y$

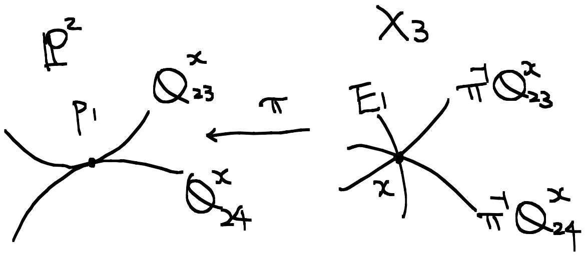
Thus, $X_3 \xrightarrow{|-K_{X_3}|} \mathbb{P}^3$ separate points

② $x \in$ exceptional set $\bigcup_{i=1}^{6} E_i$

then $Q_i \cup \pi^{-1}(\overline{p_i \pi(x)})$ has different tangents at x.

Assume that $x \in E_1$,

Q^x_{23}, Q^x_{24} are tangent at $p_1 = \pi(x)$ of order 2.



So $\pi^{-1}Q^x_{23}, \pi^{-1}Q^x_{24}$ has different tangents at x

$\therefore$ $X_3 \xrightarrow{|-K_{X_3}|} \mathbb{P}^3$ separates tangents.

- Assume $X_3 \xhookrightarrow{|-K_{X_3}|} \mathbb{P}^3$ is of degree d.

Then $(-K_{X_3})^2 = 3$ by canonical formula of blow-ups.

$X_3 \xhookrightarrow{|-K_{X_3}|} \mathbb{P}^3$ $\quad \|$

$$\left(\mathcal{O}(1)|_{X_3}\right)^2 = d$$

$\deg X_3 = d$

- $h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)) = \binom{2+4}{2} = 15$

Claim: $h^0(\mathbb{P}^4, (\mathcal{O}_{\mathbb{P}^4}(2))(-X_4)) \geq 2$

$$0 \to \mathcal{O}_{\mathbb{P}^4}(2)(-X_4) \to \mathcal{O}_{\mathbb{P}^4}(2) \to \mathcal{O}_{X_4}(2) \to 0$$

It suffices to show that $h^0(X_4, \mathcal{O}_{X_4}(2)) \leq 13$

$C = X_4 \cap \underset{\text{generic hyperplane}}{\underline{H}}$ smooth

$$0 \to \mathcal{O}_{X_4}(H) \to \mathcal{O}_{X_4}(2H) \to \mathcal{O}_C(2H) \to 0$$

$$\therefore\ h^0(X_4, \mathcal{O}_{X_4}(2)) \leq h^0(X_4, \mathcal{O}_{X_4}(H)) + h^0(C, \mathcal{O}_C(2H))$$

$$\leq \underbrace{h^0(X_4, \mathcal{O}_{X_4})}_{1} + h^0(C, \mathcal{O}_C(H)) + h^0(C, \mathcal{O}_C(2H))$$

$h^0(C, \mathcal{O}_C(H)) = \deg \mathcal{O}_C(H) - 1 + 1 = H^2 = 4$

$h^0(C, \mathcal{O}_C(2H)) = \deg \mathcal{O}_C(2H) - 1 + 1 = 8$

Therefore, $X_4 \subseteq Q_1 \cap Q_2$, Q_i quadrics

$\Downarrow$ X_4 irreducible surface, $\#(X_4 \cap H_1 \cap H_2) = 4$

$X_4 = Q_1 \cap Q_2$

Conversely, all smooth cubic surfaces, surfaces from intersection of quadrics all come from this construction.

Proposition 3. $X \subseteq \mathbb{P}^3$ smooth cubic surface, then $X = Bl_{P_1, \cdots, P_6} \mathbb{P}^2$.

Lemma 1: X contains a line.

pf: $\mathcal{P} = |\mathcal{O}_{\mathbb{P}^3}(3)| \cong \mathbb{P}^{\binom{3+3}{3}-1=19}$

$G_4 = Gr(\mathbb{P}^1, \mathbb{P}^3)$ of dimension $(1+1)\times((3+1)-(1+1)) = 4$

quadrics in $\mathbb{P}^5$ via Plücker embedding

$Z = \{ (l, S) \in G_4 \times \mathcal{P} \mid l \subseteq S \}$ incidence variety

proper

$$\begin{array}{ccc} & Z & \\ \swarrow^{p_1} & & \searrow^{p_2} \\ G_4 & & \mathcal{P} \ni X \end{array}$$

$\circlearrowright G_4$

PGL(4) transitive

(x, y, z, w) coordinates on $\mathbb{P}^3$

If $l = \{ z = w = 0 \} \subseteq X$, defining equation of X

will have vanishing coefficients for x^3, x^2y, xy^2, y^3.

$\Longrightarrow p^{-1}(l)$ irreducible of dimension $\dim \mathcal{P} - 4$

$\therefore \dim Z = \dim p^{-1}(l) + \dim G_4 = \dim \mathcal{P}$

If p_2 is NOT surjective, then $X_3 \in Im\, p_2$.

$p_2^{-1}(X_3)$ would be at least 1-dim'l.

Notice that • lines in del Pezzo surfaces are (-1)-curves

$\because \underset{-2}{\deg K_C} = (K_{X_d} + C).C = \underset{-1}{-H.C} + \underset{-1}{C^2}$

• Only (-1)-curves in X_3 are

– exceptional curves E_i 6

– $\pi^{-1}(\overline{p_i p_j})$ $\binom{6}{2} = 15$

– $\pi^{-1}(Q)$, Q: conic passing throug five of $p_1, \cdots, p_6$ $\qquad \binom{6}{5} = 6$

So $\underline{27 \text{ lines on } X_3}$!!

Lemma 2. $l \subseteq X$ line. Then $\exists$ 5 pairs of disjoint lines intersecting l in X.

pf: Let $\{P_\lambda\}_{\lambda \in \mathbb{P}^1}$ be the planes in $\mathbb{P}^3$ containing l.

$P_\lambda \cap X = l \cup C_\lambda$, C_λ conic

C_λ can't degenerate to a double line nor $l \subseteq C_\lambda$

Otherwise, equation of X will be of the form

$LQ + MN^2 = 0$, L, M, N linear. Q quadratic

Then X is singular at $\{L = N = Q = 0\} \rightarrow \times$

WLOG, assume that $l = \{z = w = 0\}$

$\Rightarrow$ X defined by

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0 \qquad (*)$$

$\overset{1}{A}, \overset{1}{B}, \overset{1}{C}, \overset{2}{D}, \overset{2}{E}, \overset{3}{F}$ homog. polynomials$^{\text{deg}}$ in z, w.

$$C_\lambda = \begin{cases} (*) \\ z = \lambda w \end{cases} \text{ is singular iff } \begin{vmatrix} \overset{1}{A} & \overset{1}{B} & \overset{2}{D} \\ \overset{1}{B} & \overset{1}{C} & \overset{2}{E} \\ \overset{2}{D} & \overset{2}{E} & \overset{3}{F} \end{vmatrix} = 0 \qquad (**)$$

quintic in z, w

$$\begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix} = 0 \Leftrightarrow \begin{pmatrix} A & B & D \\ B & C & E \\ D & E & F \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & A' & 0 \\ 0 & 0 & B' \end{pmatrix}$$

A conic is degenerate iff it is union of lines.

Claim : (**) has no repeated root.

Assume that $z=0$ is a root of (**)

$s \in C_0$ the singular point of the conic.

① $s \notin l$, then can assume that $C_0 = \{xy=0\}$

$\Rightarrow$ all coefficients in (*) are divided by z except B.

s is a smooth point of X

$\Rightarrow$ F can't be divided by z^2.

Otherwise X is singular at $s = (\underset{x}{0}, \underset{y}{0}, \underset{z}{0}, \underset{w}{1})$

② $s \in l$, then can assume that $C_0 = \{x^2 - w^2 = 0\}$

Similarly, s smooth point on X implies that

z is a single root.

Therefore, there are $\underline{5 \text{ pairs of lines}}$ intersecting l.

$\{d_i, d_i'\}$ s.t $d_i \cup d_i' = C_{\lambda_i}$. $i=1,\dots,5$

Then d_1, d_2 is disjoint.

Otherwise, d_1, d_2, l is contained in the same plane P_λ

and $d_1 \cup d_2 = C_{\lambda_1} = C_{\lambda_2}$ $\longrightarrow\!\!\!\!\!\times$

Proof of Proposition 3:

l, l' disjoint lines in X

$l \times l' \overset{\phi}{\dashrightarrow} X$

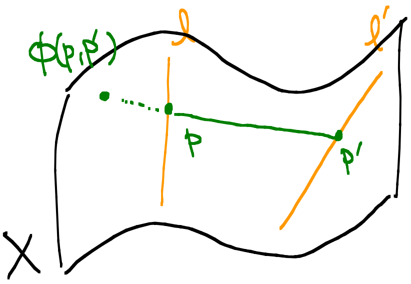
$(p, p') \longmapsto$ the third point of $\overline{pp'} \cap X$

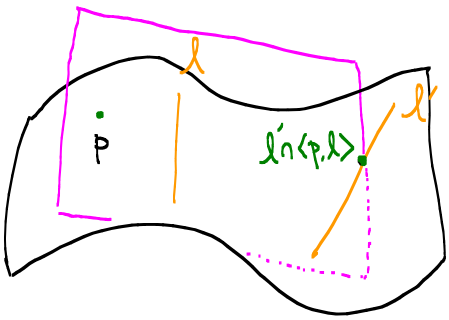
not defined if $\overline{pp'}$ tangent to X

$X \overset{\psi}{\dashrightarrow} l \times l'$

$p \longmapsto \left(l \cap \langle p, l' \rangle,\ l' \cap \langle p, l \rangle \right)$

$\notin$
$l \cup l'$





- ψ, ϕ inverse of each other.
- ψ is actually a morphism.

  If $p \in l$, then replace $\langle p, l \rangle$ be the tangent plane at p.

  Contracting lines intersecting l, l'.

  Lemma 2 $\Longrightarrow$ Five pairs of lines $\{d_i, d_i'\}$ intersecting l.

  d_i, d_i', l colinear in plane P_i $\quad P_i \cap X = d_i \cup d_i' \cup l$

  $P_i \cap l' = pt \in$ one of d_i, d_i'

  Otherwise, d_i, d_i', l' colinear & $l \cap l' \neq \phi$

  Thus, ψ contracts five lines.

  or $\quad X =$ Blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ at 5 points

  $\quad\quad = $ Blow up of $\mathbb{P}^2$ at 6 points

  $X \underset{\text{cubic}}{\overset{|-K_X|}{\hookrightarrow}} \mathbb{P}^3 \Longrightarrow X$ is dP_3.

Proposition 4. $X = Q_1 \cap Q_2$, $Q_i \subseteq \mathbb{P}^4$ quadric

$\Rightarrow X$ is dP_4

pf: 1) X contains only finitely many lines.

- $E \subseteq X$ line, H: hyperplane in $\mathbb{P}^4$

$$E.H = 1 \underset{-K_X = H}{\Rightarrow} -K.E = 1 \underset{\text{adjunction}}{\Longrightarrow} E^2 = -1$$

- E, E' distinct lines $\Longrightarrow E.E' = 0$ or 1

$\therefore [E], [E']$ different classes in $NS(X)$.

- Given line $l, E, E' \subseteq X$ st $l \cap E$, $l \cap E' \neq \phi$

then $E \cap E' = \phi$ otherwise the plane containing $l, E, E' \subseteq X$

$\Rightarrow$ Only finitely many lines in X intersecting l.

otherwise, they are all disjoint & linear independent in $NS(X)$

$\Rightarrow$ Only finitely many lines in X.

again from finiteness of $NS(X)$.

2) $p \in X$ does not fall on any line in X.

$$\begin{tikzcd} & X \setminus \{p\} \arrow[r, dashed] \arrow[dl, phantom, "\supseteq"] \arrow[dr, dotted] & \mathbb{P}^3 \\ X & & X' \arrow[u] \\ Bl_p X \arrow[u] \arrow[urr, "\cong"'] & & \end{tikzcd}$$

projection from p.

dP_3

Sends generic hyperplanes to hyperplanes

$\cup$ $\qquad\qquad$ $\cup$

smooth anti-canonical divisors $\qquad$ smooth cubics

$\therefore X'$ Cubic surface

Any trisecant of X is contained in Q_i. $i = 1, 2$

$Q_i \cap$ line $=$ 2 points or the whole line

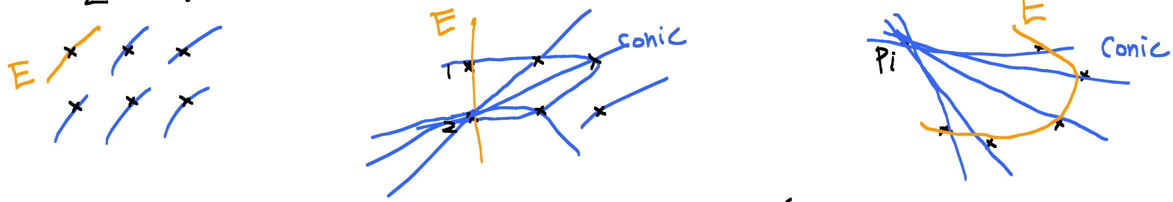
Thus, any trisecant of X is contained in X.

$\Longrightarrow$ no trisecant of X passing through p.

$$\therefore\ \underset{\underset{E}{\cup}}{Bl_p X} \cong X' = Bl_{p_1,\dots,p_6}\mathbb{P}^2$$

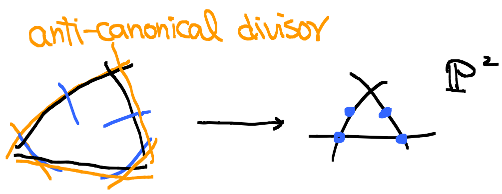
E exceptional divisor

• $E = E_i$ • $E = \overline{p_1p_2}$ • E conic not passing p_i



After blow down 6 disjoint (-1) curves, one reaches a smooth minimal surface w/ $\mathrm{Pic} \cong \mathbb{Z}$. $\rightsquigarrow \mathbb{P}^2$.

ex. dP_5



$\wr\wr$

$\underline{(1,2)} \subseteq \mathbb{P}^1_{(u,v)} \times \mathbb{P}^2_{(x,y,z)} \to Q_1(x,y,z)u + Q_2(x,y,z)v = 0$

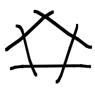
$\downarrow$

$\mathbb{P}^2$

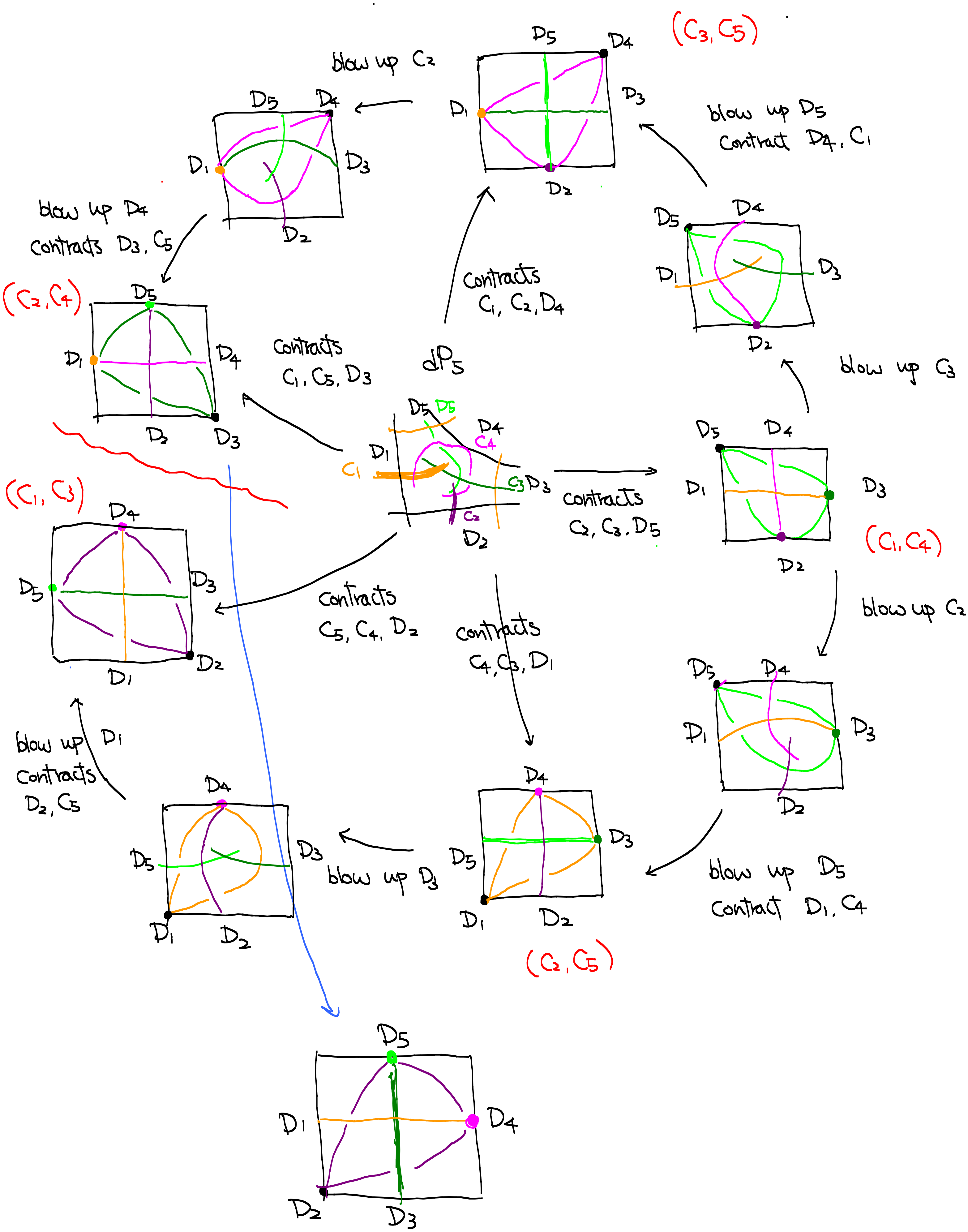
it is the blow up of $\underline{Q_1 \cap Q_2}$

4 points

There are $4 + C^4_2 = 10$ (-1)-curves.

$dP_5 \setminus$  is union of 5 $\underline{(\mathbb{C}^*)^2}$

1. 5 cluster chart of A_2 cluster algebra
2. Maurey–Cartan spaces of 5 monotone Lagrangian torus in dP_5



dP_5
blow up C_2
(C_3, C_5)
blow up D_5 contract D_4, C_1
blow up D_4 contracts D_3, C_5
(C_2, C_4)
contracts C_1, C_5, D_3
contracts C_1, C_2, D_4
blow up C_3
contracts C_2, C_3, D_5
(C_1, C_4)
(C_1, C_3)
contracts C_5, C_4, D_2
contracts C_4, C_3, D_1
blow up C_2
blow up D_1 contracts D_2, C_5
blow up D_3
blow up D_5 contract D_1, C_4
(C_2, C_5)

ex. $X_2 = dP_2$, $h^0(X_2, -K_{X_2}) = 3$

- $X_2 \xrightarrow{|-K_{X_2}|} \mathbb{P}^2$ finite morphism   quasi-finite + proper $\Longrightarrow$ finite
- $2 = (-K_{X_2})^2 = (\pi^* H)^2 = d \cdot H^2 = d$, $d =$ degree of $\Phi_{|-K_{X_2}|}$

$$\begin{array}{c} X_2 \;\; (x, z_2) \supseteq \widetilde{R} = \{x = 0\} \\ \pi = |-K_{X_2}| \Big\downarrow 2:1 \\ \mathbb{P}^2 \supseteq (4) \\ (z_1, z_2) \end{array}$$

locally $x^2 = z_1$

ramification locus $R = \{z_1 = 0\}$

$\pi^*(f\, dz_1 \wedge dz_2) = (\pi^* f) + \widetilde{R}$

$\therefore K_{X_2} = \pi^* K_{\mathbb{P}^2} + \widetilde{R}$

$= \pi^*(K_{\mathbb{P}^2} + \frac{1}{2} R) = \pi^*(-H)$

$\therefore$ R smooth quartic.

There are $\underline{56}$ (-1)-curves in dP_2,

$7 + \binom{7}{2} + \binom{7}{5} + \underline{\binom{7}{1}}$ cubic passing throug 7 points w/ a node one of of them

which map in pairs to 28 bitangent of the quartic.

- $C = (-1)$-curve , $K_{X_2} \cdot C = -1$

$\Downarrow$

$1 = \pi^* H \cdot C = H \cdot \pi_* C \Rightarrow C \xrightarrow{1:1} \pi(C)$ line in $\mathbb{P}^2$

- $\widetilde{R} \cdot C = \pi^*(2H) \cdot C = 2$

correspond to 2 points $\pi(C)$ tangent to R

$X_2 \xhookrightarrow{|-2K_{X_2}|} \mathbb{P}^6$

It suffices to prove that
① $h^0(X_2, -2K_{X_2}) = 7$
② $|-2K_{X_2}|$ separate points

$2x\,dx \wedge dz_2$

① $h^0(X_2, -2K_{X_2}) = h^0\left(X_2, \underline{+6\pi^* H - 2\sum_{i=1}^{7} E_i}\right) = 28 - 3\times 7 = 7$

generically proper transform of degree 6 curves in $\mathbb{P}^2$ w/ simple nodes on p_i $\qquad \binom{6+2}{2} = 28$

Assume the deg 6 curve $f(x,y)=0$

$\frac{\partial f}{\partial x}(p_i) = \frac{\partial f}{\partial y}(p_i) = f(p_i) = 0$ imposes 3 linear relation on each node

② Those elements in $|-2K_{X_2}|$ do NOT separate points
$\qquad\qquad\qquad\quad |-K_{X_2}|$

are double cover of conics in $\mathbb{P}^2$ of 5-dim'l family
$\qquad\qquad\qquad\quad$ lines $\qquad\qquad\qquad$ parametrized by $Gr(\mathbb{P}^1, \mathbb{P}^3)$